

SINGULARITIES ON THE BASE OF A FANO TYPE FIBRATION

CAUCHER BIRKAR

ABSTRACT. Let $f: X \rightarrow Z$ be a Mori fibre space. McKernan conjectured that the singularities of Z are bounded in terms of the singularities of X . Shokurov generalised this to pairs: let (X, B) be a klt pair and $f: X \rightarrow Z$ a contraction such that $K_X + B \sim_{\mathbb{R}} 0/Z$ and that the general fibres of f are Fano type varieties; adjunction for fibre spaces produces a discriminant divisor B_Z and a moduli divisor M_Z on Z . It is then conjectured that the singularities of $(Z, B_Z + M_Z)$ are bounded in terms of the singularities of (X, B) . We prove Shokurov conjecture when $(F, \text{Supp } B_F)$ belongs to a bounded family where F is a general fibre of f and $K_F + B_F = (K_X + B)|_F$.

1. INTRODUCTION

We work over an algebraically closed field k of characteristic zero. Let X be a variety with klt singularities and $f: X \rightarrow Z$ a K_X -negative extremal contraction. When f is a divisorial contraction, we can check that the singularities of Z are as good as the singularities of X . More precisely, in terms of log discrepancies we have

$$a(E, X, 0) \leq a(E, Z, 0)$$

for every prime divisor E on birational models of X, Z . If f is a flipping contraction, and if $f^+: X^+ \rightarrow Z$ is the positive side of the flip, then singularities of X^+ are as good as the singularities of X . This is important since if we want to prove a statement about X we can often translate it into a similar statement about Z or X^+ (eg, finite generation).

If f is not birational, one would still like to understand the singularities on Z although this is much more complicated. Beside being an interesting problem on its own, it is also important for inductive arguments. McKernan conjectured that the singularities of Z are bounded in terms of the singularities of X , that is:

Conjecture 1.1 ($M_{d,\epsilon}$). *Let d be a natural number and $\epsilon > 0$ a real number. Then, there is a real number $\delta > 0$ depending on d, ϵ satisfying the following: let $f: X \rightarrow Z$ be a K_X -negative extremal contraction such that*

- X is ϵ -lc of dimension d and \mathbb{Q} -factorial, and
- $\dim X > \dim Z$.

Then Z is δ -lc.

Date: October 10, 2012.
2010 MSC: 14E30, 14J17.

See 2.1 for the definition of ϵ -lc singularities. When $d = 1$ or $d = 2$, the conjecture is trivial since Z would be a smooth curve or just a point. Mori and Prokhorov [16, Theorem 1.2.7] proved the conjecture for $d = 3$ and $\epsilon = 1$ but with X having terminal singularities: in this case one can take $\delta = 1$. Much more recently, Alexeev and Borisov [2] proved the conjecture for toric morphisms of toric varieties.

Shokurov generalised the conjecture to the setting of pairs. Let $f: X \rightarrow Z$ be a contraction of normal varieties, and (X, B) klt such that $K_X + B \sim_{\mathbb{R}} 0/Z$. By a construction of Kawamata [12][13] we may write

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$$

where B_Z is called the *discriminant part* and M_Z is called the *moduli part*. The discriminant part is canonically determined as a Weil \mathbb{R} -divisor by the singularities of (X, B) and the fibres over codimension one points of Z ; the moduli part is then automatically determined as an \mathbb{R} -linear equivalence class, in particular, M_Z may be represented by many different Weil \mathbb{R} -divisors. See 2.6 for more details.

We are ready to state a refined version of Shokurov's conjecture.

Conjecture 1.2 ($S_{d,\epsilon,\mathcal{P}}$). *Let d be a natural number, $\epsilon > 0$ a real number, and \mathcal{P} a set of couples. Then, there is a real number $\delta > 0$ depending on d, ϵ, \mathcal{P} satisfying the following: let (X, B) be a pair and $f: X \rightarrow Z$ a contraction such that*

- (X, B) is ϵ -lc of dimension d ,
 - $K_X + B \sim_{\mathbb{R}} 0/Z$,
 - the general fibres F of f are of Fano type,
 - $(F, \text{Supp } B_F)$ is isomorphic in codimension one with some $(F', D_{F'}) \in \mathcal{P}$
- where $K_F + B_F = (K_X + B)|_F$.

Then, we can choose an \mathbb{R} -divisor $M_Z \geq 0$ representing the moduli part so that $(Z, B_Z + M_Z)$ is δ -lc.

See 2.1, 2.3, and 2.8 for the definition of general fibres, Fano type varieties, couples and their boundedness. A couple is essentially a pair but with no condition on singularities except normality. Note that unlike in Conjecture 1.1, f is allowed to be a divisorial contraction, a flipping contraction, or a fibre type contraction. Also note that we are not assuming X, Z to be projective although f is projective. Mori and Prokhorov [17, Theorem 1.1] prove a result on weak del Pezzo fibrations in dimension 3 which is closely related to the conjecture when $d = 3$, $\epsilon = 1$, $\dim Z = 1$, and $-K_F$ is nef and big.

Now we come to the main theorem of this paper.

Theorem 1.3. *Shokurov Conjecture $S_{d,\epsilon,\mathcal{P}}$ holds if \mathcal{P} is a bounded family of couples.*

To prove the theorem, we use a recent result of Hacon, McKernan, and Xu [10, Theorem 1.3] on volumes of big log divisors. In view of the theorem, it is natural to consider interesting cases of bounded \mathcal{P} (Corollaries 1.4 and 1.5) and

to try to reduce the conjecture to the theorem when \mathcal{P} is not bounded (proof of Corollary 1.6 and Remark 4.3).

Let d be a natural number and $\epsilon, \lambda > 0$ be real numbers. Consider the pairs (F, B_F) satisfying:

- (F, B_F) is ϵ -lc and of dimension $\leq d - 1$,
- $K_F + B_F \sim_{\mathbb{R}} 0$,
- F is of Fano-type,
- each non-zero coefficient of B_F is $\geq \lambda$.

Let \mathcal{R} be the set of the couples $(F, \text{Supp } B_F)$. It is expected that \mathcal{R} is a bounded family. This boundedness is known when $d \leq 3$ (see Theorem 4.1). Actually, if one tries to prove the boundedness in any dimension, then Theorem 1.3 appears naturally (see Remark 4.2 for a discussion on this).

Corollary 1.4. *Conjecture $S_{d,\epsilon,\mathcal{R}}$ holds for the above data d, ϵ, \mathcal{R} when $d \leq 3$.*

Under some extra assumptions, the boundedness of \mathcal{R} is known in any dimension. More precisely: let d be a natural number, $\epsilon > 0$ a real number, and $\Lambda \subset [0, 1]$ a finite set of real numbers. Consider the pairs (F, B_F) satisfying:

- (F, B_F) is ϵ -lc and of dimension $\leq d - 1$,
- $K_F + B_F \sim_{\mathbb{R}} 0$,
- F is projective and $-K_F$ is ample, i.e. F is a Fano variety,
- the coefficients of B_F belong to Λ .

Let \mathcal{Q} be the set of the couples $(F, \text{Supp } B_F)$. By [10, Corollary 1.7], \mathcal{Q} is a bounded family.

Corollary 1.5. *Conjecture $S_{d,\epsilon,\mathcal{Q}}$ holds for the above data d, ϵ, \mathcal{Q} .*

For surfaces we can verify $S_{d,\epsilon,\mathcal{P}}$ without boundedness assumptions:

Corollary 1.6. *Conjecture $S_{2,\epsilon,\mathcal{P}}$ holds. More generally: Conjecture $S_{d,\epsilon,\mathcal{P}}$ holds for those (X, B) and $f: X \rightarrow Z$ with $d \leq \dim Z + 1$.*

We say a few words about the proof of Theorem 1.3. The difficult part of the theorem is to deal with the discriminant part B_Z since by applying a result of Ambro [3] we can control the moduli part M_Z (actually we have to understand the discriminant b-divisor \mathcal{B}_Z rather than just B_Z). By taking hyperplane sections of Z one can reduce the problem to the case $\dim Z = 1$. Here one is mainly concerned about bounding the multiplicities of each fibre of some fibration birational to f . Mori and Prokhorov [17, Theorem 1.1] do this by using the orbifold Riemann-Roch theorem for varieties of dimension 3 with terminal singularities. Unfortunately, this approach does not work in higher dimensions. Instead of Riemann-Roch, we use the boundedness of volumes of big log divisors [10, Theorem 1.3].

Shokurov has another approach to the theorem: as far as I understand he is trying to construct a compactified coarse moduli space for the log general fibres and then recover information about f from the moduli space. Our approach is more direct and it does not rely on the existence of such a moduli space. However, the techniques developed in this paper might in fact be useful to construct

such moduli spaces.

Acknowledgements: This work was supported by a grant from Leverhulme. Thanks to Yuri Prokhorov for useful discussions on his results with Mori.

2. PRELIMINARIES

2.1. Pairs. We work over an algebraically closed field k of characteristic zero: all the varieties and schemes are over k unless stated otherwise. A *sub-pair* (X, B) consists of a normal quasi-projective variety X and a sub-boundary B , that is, an \mathbb{R} -divisor on X with coefficients in $(-\infty, 1]$ such that $K_X + B$ is \mathbb{R} -Cartier. For a prime divisor D on some birational model of X with a nonempty centre on X , $a(D, X, B)$ denotes the log discrepancy. We say that (X, B) is ϵ -lc if $a(D, X, B) \geq \epsilon$ for every prime divisor D on birational models of X (if $B = 0$, we also say that X is ϵ -lc). This is equivalent to the following: let $g: Y \rightarrow X$ be any projective birational morphism from a normal variety Y and write $K_Y + B_Y := g^*(K_X + B)$; then every coefficient of B_Y is $\leq 1 - \epsilon$.

We call a sub-pair (X, B) lc if it is 0-lc. We call it *klt* if it is ϵ -lc for some $\epsilon > 0$. A sub-pair (X, B) is called a *pair* if the coefficients of B are non-negative: in this case we call B a boundary. We refer to [14] for standard definitions and results on singularities of pairs and the log minimal model program.

Let (X, B) be a lc sub-pair and $M \geq 0$ an \mathbb{R} -Cartier divisor. The *lc threshold* of M with respect to (X, B) is the largest real number t so that $(X, B + tM)$ is lc.

A *contraction* $f: X \rightarrow Z$ is a projective morphism of quasi-projective varieties with $f_*\mathcal{O}_X = \mathcal{O}_Z$. A general fibre of f is a fibre over a closed point belonging to some fixed open set $U \subset Z$. In practice, U is understood from the context and we might shrink it without mention. If (X, B) is a pair, a *log general fibre* of (X, B) and f is as (F, B_F) where F is a general fibre of f and $K_F + B_F = (K_X + B)|_F$.

2.2. Minimal models and Mori fibre spaces. Let (X, B) be a lc pair, (Y, B_Y) a \mathbb{Q} -factorial dlt pair, $X \rightarrow Z$ and $Y \rightarrow Z$ contractions, $\phi: X \dashrightarrow Y/Z$ a birational map such that ϕ^{-1} does not contract divisors, and $B_Y = \phi_*B$. Moreover, assume that

$$a(D, X, B) \leq a(D, Y, B_Y)$$

for any prime divisor D on birational models of X and assume that the strict inequality holds for any prime divisor D on X which is exceptional/ Y . We say that (Y, B_Y) is a *log minimal model* of (X, B) over Z if $K_Y + B_Y$ is nef/ Z . On the other hand, we say that (Y, B_Y) is a *Mori fibre space* of (X, B) over Z if there is a $K_Y + B_Y$ -negative extremal contraction $Y \rightarrow Y'/Z$ such that $\dim Y' < \dim Y$. These definitions follow the traditional definitions of log minimal models and Mori fibre spaces. The corresponding definitions in [6] are more general but we do not need such generality in this paper.

Assume that (X, B) is a \mathbb{Q} -factorial klt pair and $f: X \rightarrow Z$ a contraction with $K_X + B$ or B big/ Z . Also assume that $C \geq 0$, $(X, B + C)$ is klt, and $K_X + B + C$ is nef/ Z . Then by [7], any LMMP/ Z on $K_X + B$ with scaling of C ends up with a log minimal model or a Mori fibre space over Z .

2.3. Fano type varieties. A projective variety X is said to be of *Fano type* if there is a boundary C such that (X, C) is a klt pair and $-(K_X + C)$ is ample. Note that if there is another boundary B with $K_X + B \sim_{\mathbb{R}} 0$, then B is big because $-K_X$ is big.

Lemma 2.4. *Assume that $\phi: X \dashrightarrow X'$ is an isomorphism in codimension one between normal projective varieties. If X is of Fano type, then X' is also of Fano type.*

Proof. There is a boundary C such that (X, C) is a klt pair and $-(K_X + C)$ is ample. There is an \mathbb{R} -Cartier divisor D such that $(X, C + D)$ is klt and $K_X + C + D \sim_{\mathbb{R}} 0$. Obviously, D is ample. Now, $(X', C' + D')$ is klt, $K_{X'} + C' + D' \sim_{\mathbb{R}} 0$, and D' is big where C' denotes the birational transform of C (similar notation for the other divisors). We can write $D' \sim_{\mathbb{R}} A' + G'$ where A' is ample and $G' \geq 0$. Thus,

$$(X', C' + (1 - t)D' + tA' + tG')$$

is klt and

$$K_{X'} + C' + (1 - t)D' + tA' + tG' \sim_{\mathbb{R}} 0$$

if $t > 0$ is sufficiently small. Put $\Delta' = C' + (1 - t)D' + tG'$. Then, (X', Δ') is klt and $-(K_{X'} + \Delta')$ is ample hence X' is also of Fano type. \square

2.5. B-divisors. Let X be a normal variety. An \mathbb{R} -*b-divisor* \mathcal{D} on X is a collection of \mathbb{R} -divisors D_Y for each birational contraction $Y \rightarrow X$ so that if we have a birational contraction $\pi: Y' \rightarrow Y/X$, then $\pi_* D_{Y'} = D_Y$. In birational geometry, one often has to consider all the resolutions of a variety X in order to understand properties of a divisor on X . This naturally leads to b-divisors which were defined by Shokurov [19][21].

2.6. Adjunction for fibre spaces. Let $f: X \rightarrow Z$ be a contraction of normal varieties, and (X, B) a klt sub-pair such that $K_X + B \sim_{\mathbb{R}} 0/Z$, that is, we have an equivalence $K_X + B \sim_{\mathbb{R}} f^*N$ for some \mathbb{R} -Cartier \mathbb{R} -divisor N . By a construction of Kawamata [12][13] we have a decomposition

$$N \sim_{\mathbb{R}} K_Z + B_Z + M_Z$$

where B_Z is defined using the singularities of (X, B) and of the fibres of f . The part B_Z is called the *discriminant part* and the part M_Z is called the *moduli part*. More precisely, B_Z is defined as follows: for each prime divisor D on Z , let t be the lc threshold of f^*D over the generic point of D , with respect to the pair (X, B) ; then let $(1 - t)$ be the coefficient of D in B_Z . The moduli part M_Z is then determined as an \mathbb{R} -linear equivalence class.

Consider a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\tau} & X \\ \downarrow f' & & \downarrow f \\ Z' & \xrightarrow{\sigma} & Z \end{array}$$

in which X', Z' are normal, σ, τ are birational contractions, and f' is a contraction. Let $K_{X'} + B' := \tau^*(K_X + B)$. Using the relation $K_{X'} + B' \sim_{\mathbb{R}} 0/Z'$, we can similarly define a decomposition

$$\sigma^*N \sim_{\mathbb{R}} K_{Z'} + B_{Z'} + M_{Z'}$$

which satisfies $B_Z = \sigma_*B_{Z'}$ and $M_Z \sim_{\mathbb{R}} \sigma_*M_{Z'}$. Putting all the $B_{Z'}$ together for all the possible Z' determines an \mathbb{R} -b-divisor \mathcal{B}_Z . We could also choose the $M_{Z'}$ consistently so that putting all the $M_{Z'}$ together we get an \mathbb{R} -b-divisor \mathcal{M}_Z .

Now assume that X, Z are projective and that B is effective with rational coefficients. Kawamata [12][13] showed that if Z' is a sufficiently high resolution then $M_{Z'}$ is a nef \mathbb{Q} -divisor. Following ideas of Kawamata [11], Ambro [3] proved that, perhaps after replacing Z' with a higher resolution, $M_{Z'}$ satisfies a pullback property: for any other resolution $\pi: Z'' \rightarrow Z'$ we have $M_{Z''} \sim_{\mathbb{Q}} \pi^*M_{Z'}$. Moreover, he showed that $M_{Z'}$ is the pullback of a nef and big divisor under some contraction $Z' \rightarrow T$. We call such a Z' an *Ambro model*.

2.7. Volume of divisors. Let X be a normal projective variety of dimension d and D an integral divisor on X . The *volume* of D denoted by $\text{vol}(D)$ is defined as

$$\text{vol}(D) = \limsup_{m \rightarrow +\infty} \frac{h^0(\mathcal{O}_X(mD))}{m^d/d!}$$

In some places we also use the notation $\text{vol}(\mathcal{O}_X(D))$ instead of $\text{vol}(D)$.

If D is a \mathbb{Q} -divisor, we can define $\text{vol}(D) = \frac{1}{m^d} \text{vol}(mD)$ for some $m > 0$ with mD being integral. If D is not big, then it is obvious that $\text{vol}(D) = 0$. If D is nef, then $\text{vol}(D) = D^d$ which follows from the Riemann-Roch theorem. Ein-Lazarsfeld-Mustařă-Nakamaye-Popa [15] treat this topic in detail.

Assume that Ω is a set of rational numbers satisfying the descending chain condition (DCC). Hacon-McKernan-Xu [10, Theorem 1.3] (see also [9]) proved that there is a number $\theta > 0$ depending only on d and Ω such that: if (X, B) is a projective lc pair of dimension at most d , if the coefficients of B belong to Ω , and if $K_X + B$ is big, then $\text{vol}(K_X + B) \geq \theta$. We will apply this in the proof of Proposition 3.1.

2.8. Couples and bounded families. A *couple* (F, D_F) consists of a normal projective variety F and a divisor D_F on F whose coefficients are all equal to 1, i.e. D_F is a reduced divisor. The reason we call (F, D_F) a couple rather than a pair is that we are concerned with D_F rather than $K_F + D_F$ and we do not want to assume that $K_F + D_F$ is \mathbb{Q} -Cartier or that it has nice singularities.

Two couples (F, D_F) and $(F', D_{F'})$ are *isomorphic in codimension one* if there is a birational isomorphism $\phi: F \dashrightarrow F'$ which is an isomorphism in

codimension one such that $\phi_* D_F = D_{F'}$. The two couples are *isomorphic* if ϕ is an isomorphism.

We say that a set of couples \mathcal{P} is a *bounded family* if there is a projective morphism $f: S \rightarrow T$ of Noetherian schemes over k and a closed subscheme D of S such that for each $(F, D_F) \in \mathcal{P}$ there is a closed point $t \in T$ and an isomorphism between (F, D_F) and (S_t, \tilde{D}_t) where S_t, D_t are the fibres over t of the morphisms $S \rightarrow T$ and $D \rightarrow T$ respectively and $\tilde{D}_t \leq D_t$: in particular, for such t the scheme S_t is a normal projective variety and the subscheme D_t is a reduced divisor on S_t .

Lemma 2.9. *Let \mathcal{P} be a bounded family of couples. Then, there is a natural number v depending only on \mathcal{P} such that for any $(F, D_F) \in \mathcal{P}$ the volume $\text{vol}(D_F) \leq v$.*

Proof. As a general principle, the numerical invariants of a bounded family are bounded. We will give a detailed proof. Let $f: S \rightarrow T$ be as in the definition of bounded family of couples. Let T' be the reduced scheme associated to T and let $S' = S \times_T T'$ and $D' = D \times_T T'$. Let t be a closed point of T and S_t and D_t the fibres over t of the morphisms $S \rightarrow T$ and $D \rightarrow T$. These fibres are isomorphic to the corresponding fibres of the morphisms $S' \rightarrow T'$ and $D' \rightarrow T'$. Thus, we could assume from the beginning that T is a reduced scheme. Moreover, we may assume that T is affine by the Noetherian property.

Fix a closed embedding $S \rightarrow \mathbb{P}_T^n/T$ and let $\mathcal{O}_S(1)$ be the inverse image of $\mathcal{O}_{\mathbb{P}_T^n}(1)$. Since $S \rightarrow T$ is projective and T affine, there is a coherent locally free sheaf \mathcal{E} which is a direct sum of invertible sheaves of the form $\mathcal{O}_S(-m)$ with $m \gg 0$ admitting a surjective morphism $\mathcal{E} \rightarrow \mathcal{I}_D$ where \mathcal{I}_D is the ideal sheaf of D in S . This gives an exact sequence $\mathcal{E} \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_D \rightarrow 0$.

Fix a closed point $t \in T$ such that the fibre S_t is a normal variety and that D_t is a reduced divisor on S_t . Restricting the above exact sequence to S_t we get an exact sequence $\mathcal{E}_t \rightarrow \mathcal{O}_{S_t} \rightarrow \mathcal{O}_{D_t} \rightarrow 0$. In particular, we have a surjection $\mathcal{E}_t \rightarrow \mathcal{I}_{D_t}$ where \mathcal{I}_{D_t} is the ideal sheaf of D_t in S_t . We then get an injection $\mathcal{I}_{D_t}^\vee \rightarrow \mathcal{E}_t^\vee$ where for a coherent sheaf \mathcal{N} on S_t we define $\mathcal{N}^\vee := \mathcal{H}om_{\mathcal{O}_{S_t}}(\mathcal{N}, \mathcal{O}_{S_t})$. Let U be the smooth locus of S_t . Then, $\mathcal{I}_{D_t}|_U = \mathcal{O}_U(-D_t)$ hence $\mathcal{I}_{D_t}^\vee|_U = \mathcal{O}_U(D_t)$. Since \mathcal{E}_t^\vee is reflexive, the injection $\mathcal{I}_{D_t}^\vee|_U \rightarrow \mathcal{E}_t^\vee|_U$ induces an injection $\mathcal{O}_{S_t}(D_t) \rightarrow \mathcal{E}_t^\vee$.

Let

$$0 \rightarrow \mathcal{G}' \rightarrow \mathcal{G} \rightarrow \mathcal{G}'' \rightarrow 0$$

be an exact sequence of coherent locally free sheaves on S_t such that we are given an injection $\mathcal{O}_{S_t}(D_t) \rightarrow \mathcal{G}$. Then, by restricting to U one can see that either the induced morphism $\mathcal{O}_{S_t}(D_t) \rightarrow \mathcal{G}''$ is injective or $\mathcal{O}_{S_t}(D_t)$ is mapped into the kernel of $\mathcal{G} \rightarrow \mathcal{G}''$ in which case we get an injection $\mathcal{O}_{S_t}(D_t) \rightarrow \mathcal{G}'$.

By construction, \mathcal{E}_t^\vee is a coherent locally free sheaf which is a direct sum of sheaves of the form $\mathcal{O}_{S_t}(m)$ with $m \gg 0$. Applying the last paragraph we get an injection $\mathcal{O}_{S_t}(D_t) \rightarrow \mathcal{O}_{S_t}(m_t)$ for some $m_t \gg 0$ where there are only finitely many possibilities for m_t for all t as above.

Now, since T is reduced, by the generic flatness and stratification theorem, there are only finitely many possibilities for the Hilbert polynomial Φ_t of the fibre S_t . By definition, $\Phi_t(m) = \mathcal{X}(\mathcal{O}_{S_t}(m))$. On the other hand, since m_t is sufficiently large, $h^i(\mathcal{O}_{S_t}(mm_t)) = 0$ for any $i, m > 0$. Thus, $\Phi_t(mm_t) = h^0(\mathcal{O}_{S_t}(mm_t))$ for $m > 0$ and this in turn implies that the volume

$$\text{vol}(\mathcal{O}_{S_t}(m_t)) = \limsup_{m \rightarrow +\infty} \frac{h^0(\mathcal{O}_{S_t}(mm_t))}{m^{\dim S_t} / \dim S_t!} = \limsup_{m \rightarrow +\infty} \frac{\Phi_t(mm_t)}{m^{\dim S_t} / \dim S_t!}$$

depends only on Φ_t and m_t . Since there are only finitely many possibilities for Φ_t and m_t , there is a natural number v such that $\text{vol}(\mathcal{O}_{S_t}(m_t)) \leq v$ for every t as above. Thus, $\text{vol}(D_t) \leq v$ because of the injection $\mathcal{O}_{S_t}(D_t) \rightarrow \mathcal{O}_{S_t}(m_t)$. By definition, each $(F, D_F) \in \mathcal{P}$ is isomorphic to some (S_t, \tilde{D}_t) with $\tilde{D}_t \leq D_t$. Therefore,

$$\text{vol}(D_F) = \text{vol}(\tilde{D}_t) \leq \text{vol}(D_t) \leq v$$

and we are done. \square

2.10. Intersection numbers. In certain cases we will apply the following lemma to compare intersection numbers on a variety which may not be proper.

Lemma 2.11. *Let $h: Y \rightarrow Z$ be a contraction from a normal variety Y of dimension d to a smooth curve Z . Assume that L is a \mathbb{Q} -Cartier divisor on Y which is nef/ Z . Pick two distinct closed points $P, Q \in Z$ and write $h^*P = \sum m_i T_i$ and $h^*Q = \sum n_j S_j$ where T_i, S_j are the irreducible components. Then,*

$$\sum m_i (L|_{T_i})^{d-1} = \sum n_j (L|_{S_j})^{d-1}$$

Proof. Take a resolution $\phi: W \rightarrow Y$. We can write $\phi^*h^*P = \sum m_i T'_i + \sum l_k E_k$ where E_k are prime exceptional divisors of ϕ and T'_i is the birational transform of T_i . Since $\phi^*L|_{E_k}$ is not big, $(\phi^*L|_{E_k})^{d-1} = 0$. On the other hand, since the induced morphism $T'_i \rightarrow T_i$ is birational $(\phi^*L|_{T'_i})^{d-1} = (L|_{T_i})^{d-1}$. Thus,

$$\sum m_i (L|_{T_i})^{d-1} = \sum m_i (\phi^*L|_{T'_i})^{d-1} + \sum l_k (\phi^*L|_{E_k})^{d-1}$$

A similar equality holds for ϕ^*h^*Q . Therefore, by replacing X with W and L with ϕ^* we may assume that X is already smooth.

By taking compactifications and then a resolution we can assume that there is a contraction $\bar{f}: \bar{Y} \rightarrow \bar{Z}$ of smooth projective varieties such that $Y \subset \bar{Y}$, $Z \subset \bar{Z}$, and that $\bar{f}|_Y = f$. By replacing \bar{f} with f and L with its closure in \bar{Y} , we may assume that Y, Z are projective. Now, since $P - Q \equiv 0$, $h^*P \equiv h^*Q$ hence intersection with the 1-cycle L^{d-1} would be the same which implies that

$$\begin{aligned} \sum m_i (L|_{T_i})^{d-1} &= L^{d-1} \cdot (\sum m_i T_i) = L^{d-1} \cdot h^*P = L^{d-1} \cdot h^*Q \\ &= L^{d-1} \cdot (\sum n_j S_j) = \sum n_j (L|_{S_j})^{d-1} \end{aligned}$$

\square

3. BOUNDING THE DISCRIMINANT B-DIVISOR

In this section we will bound the coefficients of the discriminant b-divisor \mathcal{B}_Z that is associated to the data in Theorem 1.3. First we deal with the discriminant divisor B_Z and later we take care of the b-divisor.

Proposition 3.1. *Assume that d, ϵ, \mathcal{P} are as in Conjecture $S_{d, \epsilon, \mathcal{P}}$ and that \mathcal{P} is a bounded family of couples. Then, there is a real number $\delta > 0$ depending only on d, ϵ, \mathcal{P} such that if (X, B) and $f: X \rightarrow Z$ are as in the conjecture, then every coefficient of B_Z is $\leq 1 - \delta$.*

By taking hyperplane sections on Z we reduce the problem to the case $\dim Z = 1$.

Lemma 3.2. *Assume that Proposition 3.1 holds for the data $d - 1, \epsilon, \mathcal{P}$. Then the proposition holds for the data d, ϵ, \mathcal{P} for those (X, B) and $f: X \rightarrow Z$ with $\dim Z > 1$.*

Proof. Assume that $\delta > 0$ is as in the proposition for the data $d - 1, \epsilon, \mathcal{P}$. Let (X, B) and $f: X \rightarrow Z$ be as in the proposition for the data d, ϵ, \mathcal{P} , and assume that $\dim Z > 1$. Let D be a component of B_Z , and t the lc threshold of f^*D over the generic point of D with respect to the pair (X, B) . We will show that $t \geq \delta$ which means that the coefficient of D in B_Z is at most $1 - \delta$. By removing some codimension 2 closed subset of Z we may assume that Z is smooth. Moreover, we can assume that $(X, B + tf^*D)$ is lc whose lc centres all map onto D . By definition, $(X, B + tf^*D)$ has at least one lc centre.

Pick a general hyperplane section $H \subset Z$ (which would intersect D) and let $G = f^*H$. This ensures that $(X, B + G + tf^*D)$ is lc and that $(X, B + G)$ is ϵ -lc in codimension ≥ 2 , that is, $a(P, X, B + G) \geq \epsilon$ for any prime divisor P exceptional/ X . Letting

$$K_G + B_G = (K_X + B + G)|_G$$

we get a klt pair (G, B_G) and a contraction $g: G \rightarrow H$ which satisfy properties of the proposition for the data $d - 1, \epsilon, \mathcal{P}$: indeed, (G, B_G) is ϵ -lc of dimension $d - 1$, $K_G + B_G \sim_{\mathbb{R}} 0/H$, and the log general fibres of g are among the log general fibres of f .

By further shrinking Z around D we can assume that $D_H := D \cap H$ is irreducible. By construction, $(G, B_G + tg^*D_H)$ is lc. Moreover, if V is a lc centre of $(X, B + tf^*D)$ then $V \cap G \neq \emptyset$ because V is mapped onto D and G contains every fibre over points of $D_H \subset D$. Since V and G are both lc centres of $(X, B + G + tf^*D)$, by Ambro [4, Theorem 1.1] the intersection $V \cap G$ is a union of lc centres of $(X, B + G + tf^*D)$. Each lc centre of $(X, B + G + tf^*D)$ which sits inside G is also a lc centre of $(G, B_G + tg^*D_H)$. Thus, $V \cap G$ is a union of lc centres of $(G, B_G + tg^*D_H)$. As $V \cap G$ maps onto D_H , there is a lc centre W of $(G, B_G + tg^*D_H)$ which maps onto D_H . In particular, this means that t is the lc threshold of g^*D_H with respect to the pair (G, B_G) over the generic point of D_H . By assumptions, $t \geq \delta$. Therefore, the coefficient of D in B_Z is at most $1 - \delta$ and we are done. \square

Proof. (of Proposition 3.1) *Step 1.* Let d, ϵ, \mathcal{P} be as in the proposition where \mathcal{P} is a bounded family by assumption. We will apply induction so we can assume that the proposition holds for the data $d', \epsilon, \mathcal{P}$ if $d' < d$. Let (X, B) and $f: X \rightarrow Z$ be as in the proposition for the data d, ϵ, \mathcal{P} . By Lemma 3.2, we may assume that $\dim Z = 1$. By taking a \mathbb{Q} -factorialisation and applying Lemma 2.4 we may assume that X is \mathbb{Q} -factorial.

Fix a closed point $D \in Z$. We will find a real number $\delta > 0$ depending only on d, ϵ, \mathcal{P} such that the coefficient of D in B_Z is $\leq 1 - \delta$. We can shrink Z around D if necessary. In particular, we can assume that each component of B is either horizontal/ Z or mapped to D .

Step 2. By decreasing ϵ if necessary we may assume that ϵ is rational. Put $b = 1 - \frac{\epsilon}{2}$. Since (X, B) is ϵ -lc, and since for any prime divisor P on X we have $\epsilon \leq a(P, X, B) \leq 1$, $b > 0$. Let $\phi: W \rightarrow X$ be a log resolution of $(X, B + f^*D)$. Let $\{M_i\}$ be the set of components of ϕ^*f^*D , let $\{M'_j\}$ be the set of prime exceptional divisors of ϕ which do not belong to $\{M_i\}$, and let $\{M''_k\}$ be the set of components of the birational transform of B which do not belong to $\{M_i\}$. Define

$$\Delta_W = \sum M_i + \sum bM'_j + \sum b_k M''_k$$

where b_k is the coefficient of M''_k in the birational transform of B . Define

$$\Gamma_W = \sum M_i + \sum bM'_j + \sum bM''_k$$

By construction, (W, Δ_W) and (W, Γ_W) are both dlt, $\text{Supp } \Gamma_W = \text{Supp } \Delta_W$, $\Gamma_W - \Delta_W \geq 0$,

$$[\Gamma_W] = [\Delta_W] = \text{Supp } \phi^*f^*D,$$

each component of $\Gamma_W - \Delta_W$ is a component of the birational transform of B and horizontal/ Z , and $\Gamma_W - \Delta_W$ and $[\Gamma_W] = [\Delta_W]$ have no common components. Also, note that the coefficients of Γ_W belong to the set $\{b, 1\}$.

Write $K_W + B_W = \phi^*(K_X + B)$. Since (X, B) is ϵ -lc, each coefficient of B_W is at most $1 - \epsilon$. If Q is a component of B_W with positive coefficient, then Q is exceptional/ X or a component of the birational transform of B . In either case Q is a component of Γ_W . So from $1 - \epsilon < b$ we get $\Gamma_W - B_W \geq 0$ and that $\text{Supp } \phi^*B \subseteq \text{Supp}(\Gamma_W - B_W)$. Thus, $B_W + r\phi^*B \leq \Gamma_W$ if $r > 0$ is sufficiently small. This implies that

$$\phi^*(K_X + B + rB) \leq K_W + \Gamma_W$$

hence $K_W + \Gamma_W$ is big/ Z because $K_X + B + rB$ is big/ Z which in turn follows from the assumptions that $K_X + B \sim_{\mathbb{R}} 0/Z$ and that the general fibres of f are of Fano type (see 2.3).

Step 3. Let $\Gamma = \phi_*\Gamma_W$. We can write

$$K_W + \Gamma_W = \phi^*(K_X + \Gamma) + E_W$$

where E_W is exceptional/ X . Run the LMMP/ X on $K_W + \Gamma_W$ with scaling of some ample divisor. By [6, Theorem 3.5], after finitely many steps we get a model V on which $E_V \leq 0$. Let g be the morphism $V \rightarrow Z$. Let (G, Γ_G) be

a log general fibre of (V, Γ_V) and g , and let (F, Γ_F) be the corresponding log general fibre of (X, Γ) and f . Then,

$$K_G + \Gamma_G = \psi^*(K_F + \Gamma_F) + E_G$$

where ψ is the morphism $G \rightarrow F$ and $E_G := E_V|_G \leq 0$.

By construction, $\text{Supp } \Gamma = \text{Supp } B$ over the generic point of Z hence $\text{Supp } \Gamma_F = \text{Supp } B_F$. By assumptions, the couple $(F, \text{Supp } \Gamma_F) = (F, \text{Supp } B_F)$ is isomorphic in codimension one with some couple $(F', D_{F'}) \in \mathcal{P}$. By Lemma 2.9, the volume of $D_{F'}$ is bounded by a number v depending only on \mathcal{P} . Thus, the volume of $L_F := \text{Supp } \Gamma_F$ is also bounded by v because $\text{vol}(L_F) = \text{vol}(D_{F'})$. On the other hand, since $K_X + B \sim_{\mathbb{R}} 0/Z$, there is a rational boundary B' with the same support as B such that $K_X + B' \sim_{\mathbb{Q}} 0/Z$ (see the approximation arguments in the proof of Theorem 1.3 below) from which we get

$$K_F + \Gamma_F \leq K_F + L_F = K_F + B'_F + L_F - B'_F \sim_{\mathbb{Q}} L_F - B'_F \leq L_F$$

hence the volume of $K_F + \Gamma_F$ is bounded by v . Since $E_G \leq 0$, the volume of $K_G + \Gamma_G$ is also bounded by v .

Step 4. By construction, $\text{Supp } [\Gamma_V] = \text{Supp } g^*D$. Since $g^*D \sim 0/Z$, we can write

$$K_V + \Gamma_V \sim_{\mathbb{Q}} K_V + C_V/Z$$

where $C_V \leq \Gamma_V$, (V, C_V) is klt and $K_V + C_V$ is big/ Z . So, if we run the LMMP/ Z on $K_V + \Gamma_V$ with scaling of some ample divisor, then it terminates with a model Y on which $K_Y + \Gamma_Y$ is semi-ample/ Z . Let h denote the morphism $Y \rightarrow Z$.

Let (H, Γ_H) be a log general fibre of (Y, Γ_Y) and h , and let (G, Γ_G) be the corresponding log general fibre of (V, Γ_V) and g . By Step 3, $\text{vol}(K_G + \Gamma_G)$ is bounded. Since (Y, Γ_Y) is a log minimal model of (V, Γ_V) over Z ,

$$\text{vol}(K_H + \Gamma_H) = \text{vol}(K_G + \Gamma_G)$$

which means that $\text{vol}(K_H + \Gamma_H)$ is bounded by v as well.

Step 5. Let $Y \rightarrow Y'$ be the contraction/ Z defined by $K_Y + \Gamma_Y$. Since $K_Y + \Gamma_Y$ is big/ Z , $Y \rightarrow Y'$ is birational. Let T be a component of $\text{Supp } [\Gamma_Y] = \text{Supp } h^*D$ that is not contracted over Y' , that is,

$$K_T + \Gamma_T := (K_Y + \Gamma_Y)|_T$$

is big. Since the coefficients of Γ_Y belong to the fixed finite set $\{b, 1\}$, the coefficients of Γ_T belong to a DCC set Ω which only depend on b and d [18, Corollary 3.10]. By [10, Theorem 1.3], there is a real number $\theta > 0$ depending only on d and Ω such that $\text{vol}(K_T + \Gamma_T) \geq \theta$.

By Step 4, we can assume that $\text{vol}(K_H + \Gamma_H) \leq v$ for every fibre H of h other than h^*D . Now, write $h^*D = \sum m_i T_i$ where T_i are the irreducible components

of h^*D . By Lemma 2.11, we get the following equalities of intersection numbers

$$\begin{aligned} \text{vol}(K_H + \Gamma_H) &= (K_H + \Gamma_H)^{d-1} \\ &= ((K_Y + \Gamma_Y)|_H)^{d-1} \\ &= \sum m_i ((K_Y + \Gamma_Y)|_{T_i})^{d-1} \\ &= \sum m_i \text{vol}(K_{T_i} + \Gamma_{T_i}) \end{aligned}$$

Therefore if $K_{T_i} + \Gamma_{T_i}$ is big for some i , then

$$\text{vol}(K_H + \Gamma_H) \geq m_i \text{vol}(K_{T_i} + \Gamma_{T_i}) \geq m_i \theta$$

which implies that such m_i are bounded by $\frac{v}{\theta}$. However, we do not get any bound for m_j if $K_{T_j} + \Gamma_{T_j}$ is not big. We will try to get rid of such T_j .

Step 6. Run the LMMP/ Y' on $K_Y + \Delta_Y$ with scaling of $P_Y := \Gamma_Y - \Delta_Y$. Note that $K_Y + \Gamma_Y$ is numerically trivial on each extremal ray contracted in the process since $K_Y + \Gamma_Y \equiv 0/Y'$. The LMMP terminates for reasons similar to Step 4. In some step of the LMMP, we arrive on a model on which the birational transform of $K_Y + \Gamma_Y - rP_Y$ is semi-ample/ Y' where $r > 0$ is a small number. Replace Y with that model. Since $K_Y + \Gamma_Y$ is semi-ample/ Z , $K_Y + \Gamma_Y - rP_Y$ is also semi-ample/ Z if we take r to be sufficiently small.

Let $Y \rightarrow Y''$ be the contraction/ Z defined by $K_Y + \Gamma_Y - rP_Y$. Since r is sufficiently small, the map $Y'' \dashrightarrow Y'$ is actually a morphism. On the other hand, since, by Step 2, P_Y and $[\Gamma_Y]$ have no common components, any component of $[\Gamma_Y]$ that is contracted over Y' is also contracted over Y'' : indeed if T is a component of $[\Gamma_Y]$ that is contracted over Y' , then $(K_Y + \Gamma_Y)|_T$ is not big hence $(K_Y + \Gamma_Y - rP_Y)|_T$ is also not big; so T should be contracted over Y'' as well. Thus, by Step 5, the coefficient of any component of h''^*D is bounded by $\frac{v}{\theta}$ where h'' is the morphism $Y'' \rightarrow Z$.

Step 7. Since $K_{Y''} + \Gamma_{Y''}$ and $K_{Y''} + \Gamma_{Y''} - rP_{Y''}$ are both \mathbb{R} -Cartier, $P_{Y''}$ is also \mathbb{R} -Cartier which in turn implies that $K_{Y''} + \Delta_{Y''}$ is \mathbb{R} -Cartier as well. In particular, $K_{Y''} + \Delta_{Y''}$ is lc. Let $\pi: N \rightarrow X$ and $\mu: N \rightarrow Y''$ be a common resolution. Since (X, B) is ϵ -lc, we can write

$$K_{Y''} + B_{Y''} := \mu_* \pi^*(K_X + B)$$

where each coefficient of $B_{Y''}$ is $\leq 1 - \epsilon$. On the other hand, if we write

$$K_W + B_W = \phi^*(K_X + B)$$

as in Step 2 then $B_W + \epsilon [\Delta_W] \leq \Delta_W$ which implies that $B_{Y''} + \epsilon [\Delta_{Y''}] \leq \Delta_{Y''}$ hence

$$K_{Y''} + B_{Y''} + \epsilon [\Delta_{Y''}] \leq K_{Y''} + \Delta_{Y''}$$

Moreover, $[\Delta_{Y''}] = \text{Supp } h''^*D$.

By Steps 5 and 6, we can write $h''^*D = \sum m_i T_i$ where the m_i are all bounded by $\frac{v}{\theta}$. This implies that there is a real number $\delta > 0$ depending only on d, ϵ, \mathcal{P} such that

$$K_{Y''} + B_{Y''} + \delta h''^*D \leq K_{Y''} + \Delta_{Y''}$$

Therefore, $K_{Y''} + B_{Y''} + \delta h''^* D$ is lc which in turn implies that $K_X + B + \delta f^* D$ is lc because

$$\pi^*(K_X + B + \delta f^* D) = \mu^*(K_{Y''} + B_{Y''} + \delta h''^* D)$$

where we use the fact that $K_X + B \sim_{\mathbb{R}} 0/Z$. Note however that $B_{Y''} + \delta h''^* D$ is not necessarily effective but it is a sub-boundary. Finally, if t is the lc threshold of $f^* D$ with respect to (X, B) , then we have $t \geq \delta$. In other words, the coefficient of D in B_Z is at most $1 - \delta$ and this completes the proof of the proposition. \square

Next we bound the coefficients of the discriminant b-divisor \mathcal{B}_Z (see 2.6 for definitions). But first we need a couple of lemmas.

Lemma 3.3. *Assume that $\epsilon > \epsilon' > 0$. Let (X, B) be a \mathbb{Q} -factorial ϵ -lc pair and $f: X \rightarrow Z$ a contraction such that $K_X + B \sim_{\mathbb{R}} 0/Z$ and B is big/ Z . Then, there are normal projective varieties $X' \supseteq X$ and $Z' \supseteq Z$ and a contraction $f': X' \rightarrow Z'$ such that*

- (X', B') is \mathbb{Q} -factorial ϵ' -lc and $K_{X'} + B' \sim_{\mathbb{R}} 0/Z'$,
- $(K_{X'} + B')|_X = K_X + B$, and $f'|_X = f$.

Proof. We can compactify the morphism f to a morphism $\bar{f}: \bar{X} \rightarrow \bar{Z}$ where \bar{X} and \bar{Z} are normal projective varieties. Let $\phi: W \rightarrow \bar{X}$ be a log resolution and let $B_W = \bar{B}^\sim + (1 - \epsilon')E$ where E is the reduced exceptional divisor of ϕ , \bar{B} is the closure of B in \bar{X} , and \bar{B}^\sim is the birational transform of \bar{B} .

Run the LMMP/ \bar{X} on $K_W + B_W$ with scaling of some ample divisor. We get a model V on which $K_V + B_V$ is nef/ \bar{X} . Since (X, B) is ϵ -lc, over X , we can write $K_W + B_W \equiv G$ where G is effective and exceptional. So, the LMMP contracts every component of G over X and since X is \mathbb{Q} -factorial, $V \rightarrow \bar{X}$ is an isomorphism over X . Now, run the LMMP/ \bar{Z} on $K_V + B_V$ with scaling of some ample divisor. Since B_V is big/ \bar{Z} , the LMMP terminates with a model X' on which $K_{X'} + B'$ is semi-ample/ \bar{Z} .

Let $f': X' \rightarrow Z'/\bar{Z}$ be the contraction defined by $K_{X'} + B'$. Since $V \rightarrow \bar{X}$ is an isomorphism over X and $K_X + B \sim_{\mathbb{R}} 0/Z$, the map $V \dashrightarrow X'$ is an isomorphism over Z hence the morphism $X' \rightarrow Z'$ coincides with $X \rightarrow Z$ over Z . By construction, (X', B') and f' satisfy all the properties of the lemma. \square

Lemma 3.4. *Let (X, B) be a klt pair and $f: X \rightarrow Z$ a contraction such that $K_X + B \sim_{\mathbb{R}} 0/Z$ and B is big/ Z . Assume that D is a component of the discriminant b-divisor \mathcal{B}_Z with positive coefficient. Then, there is an extremal contraction $Z'' \rightarrow Z$ which extracts D .*

Proof. By assumptions, there is a birational contraction $g: Z' \rightarrow Z$ such that D is a component of $B_{Z'}$ with positive coefficient. By taking a \mathbb{Q} -factorialisation we may assume that X is \mathbb{Q} -factorial. By Lemma 3.3, we may assume that X, Z are projective. Assume that B has rational coefficients. We will argue as in [3]. We may assume that Z' is an Ambro model and that $(Z', B_{Z'})$ is log smooth. In particular, $M_{Z'}$ is the pullback of a nef and big divisor via some contraction $Z' \rightarrow T$. Since $(Z', B_{Z'})$ is a klt sub-pair we can pick $M_{Z'}$ so that

$M_{Z'} \geq 0$ and that $(Z', B_{Z'} + M_{Z'})$ is again a klt sub-pair. Put $M_Z = g_* M_{Z'}$. Then, $(Z, B_Z + M_Z)$ is a klt pair because

$$K_{Z'} + B_{Z'} + M_{Z'} = g^*(K_Z + B_Z + M_Z)$$

Now, $a(D, Z, B_Z + M_Z) < 1$ hence there is an extremal contraction $Z'' \rightarrow Z$ which extracts D .

If B is not a rational boundary, we can approximate B with rational boundaries (cf. Fujino-Gongyo [8, Theorem 3.1]). More precisely, using Shokurov's polytopes [19], we can find a rational boundary C sufficiently close to B such that (X, C) is klt, $K_X + C \sim_{\mathbb{R}} 0/Z$, C is big/ Z , and the coefficient of D in the discriminant b-divisor \mathcal{C}_Z is positive. Now apply the arguments above for the case of rational boundaries. \square

Proposition 3.5. *Assume that d, ϵ, \mathcal{P} are as in Conjecture $S_{d, \epsilon, \mathcal{P}}$ and that \mathcal{P} is a bounded family of couples. Then, there is a real number $\delta > 0$ depending only on d, ϵ, \mathcal{P} such that if (X, B) and $f: X \rightarrow Z$ are as in the conjecture, then every coefficient of \mathcal{B}_Z is $\leq 1 - \delta$.*

Proof. Pick $\epsilon' \in (0, \epsilon)$ and let $\delta \in (0, 1)$ be as in Proposition 3.1 for the data $d, \epsilon', \mathcal{P}$. Let (X, B) and $f: X \rightarrow Z$ be as in Conjecture $S_{d, \epsilon, \mathcal{P}}$. By taking a \mathbb{Q} -factorialisation, we may assume that X is \mathbb{Q} -factorial. This does not affect \mathcal{B}_Z . Assume that some component E of \mathcal{B}_Z has coefficient larger than $1 - \delta$. We will derive a contradiction.

By our choice of δ , every coefficient of B_Z is $\leq 1 - \delta$, so E is exceptional/ Z . Since the general fibres of f are of Fano type, B is big/ Z . By Lemma 3.4, there is an extremal contraction $g: Z'' \rightarrow Z$ such that E is the only exceptional divisor of g . Now $B_{Z''} \geq 0$ and the coefficient of E in $B_{Z''}$ is larger than $1 - \delta$. To get a contradiction with Proposition 3.1 we need to construct a suitable fibration over Z'' .

Let $\phi: W \rightarrow X$ be a log resolution so that the induced rational map $W \rightarrow Z''$ is a morphism. Let

$$\Delta_W := B^\sim + (1 - \epsilon')G$$

where B^\sim is the birational transform of B and G is the reduced exceptional divisor of ϕ . The pair (W, Δ_W) is ϵ' -lc and we can write

$$K_W + \Delta_W = \phi^*(K_X + B) + C$$

where $C \geq 0$ and $\text{Supp } C = \text{Supp } G$. In particular, if we run an LMMP on $K_W + \Delta_W$ over X (or some open subset of X), then the LMMP terminates with X (respectively with that open subset).

Let T be the graph of the rational map $X \dashrightarrow Z''$, that is, T is the closure in $X \times Z''$ of the graph of $X_0 \rightarrow Z''$ where $X_0 \subseteq X$ is the domain of $X \dashrightarrow Z''$. Since W maps to both X and Z'' we get an induced morphism $W \rightarrow T$. Let $U \subset Z$ be a non-empty open set over which $Z'' \rightarrow Z$ is an isomorphism. Run an LMMP/ T on $K_W + \Delta_W$ with scaling of some ample divisor. We end up with a model Y on which $K_Y + \Delta_Y$ is nef/ T . Since $f^{-1}U \subseteq X_0$, the morphism $T \rightarrow X$ is an isomorphism over U . So, the morphism $Y \rightarrow T$ is also an isomorphism over U by the last paragraph. Since $K_X + B \sim_{\mathbb{R}} 0/Z$, $K_Y + \Delta_Y \sim_{\mathbb{R}} 0$ over U .

Now, run an LMMP/ Z'' on $K_Y + \Delta_Y$ with scaling of some ample divisor which ends up with a model X' on which $K_{X'} + \Delta_{X'}$ is semi-ample/ Z'' because Δ_Y is big/ Z'' . Since $K_Y + \Delta_Y$ is nef over U , the LMMP does not modify Y over U , that is, $Y \dashrightarrow X'$ is an isomorphism over U .

Let $f': X' \rightarrow Z'$ be the contraction/ Z'' defined by $K_{X'} + \Delta_{X'}$. Then, the map $Z' \rightarrow Z''$ is birational. Moreover, $(X', \Delta_{X'})$ and $f': X' \rightarrow Z'$ satisfy the assumptions of Proposition 3.1 for the data $d, \epsilon', \mathcal{P}$, that is, $(X', \Delta_{X'})$ is ϵ' -lc of dimension d , $K_{X'} + \Delta_{X'} \sim_{\mathbb{R}} 0/Z'$, and the log general fibres of $(X', \Delta_{X'})$ and f' are the same as the log general fibres of (X, B) and f .

Let $\Delta_{Z'}$ be the discriminant on Z' associated to $K_{X'} + \Delta_{X'}$ and the fibration f' . By Proposition 3.1, the coefficients of $\Delta_{Z'}$ are at most $1 - \delta$. On the other hand, let $B_{Z'}$ be the discriminant on Z' associated to $K_X + B$ and the fibration f . It is enough to show that the coefficient of E in $B_{Z'}$ is not bigger than the coefficient of E in $\Delta_{Z'}$.

Let $\pi: V \rightarrow X$ and $\mu: V \rightarrow X'$ be a common resolution and let

$$M = K_{X'} + \Delta_{X'} - \mu_* \pi^*(K_X + B)$$

As mentioned above we have

$$K_W + \Delta_W - \phi^*(K_X + B) = C \geq 0$$

Now M is just the pushdown of $K_W + \Delta_W - \phi^*(K_X + B)$ via the rational map $W \dashrightarrow X'$. Therefore, $M \geq 0$. From $K_X + B \sim_{\mathbb{R}} 0/Z$ we get

$$\pi^*(K_X + B) = \mu^* \mu_* \pi^*(K_X + B)$$

and this combined with $M \geq 0$ results in

$$\begin{aligned} & \mu^*(K_{X'} + \Delta_{X'}) - \pi^*(K_X + B) \\ &= \mu^*(K_{X'} + \Delta_{X'}) - \mu^* \mu_* \pi^*(K_X + B) = \mu^* M \geq 0 \end{aligned}$$

Since the coefficient of E in $\Delta_{Z'}$ is at most $1 - \delta$, over the generic point of E the log divisor $K_{X'} + \Delta_{X'} + \delta f'^* E$ is lc. This implies that

$$\mu^*(K_{X'} + \Delta_{X'} + \delta f'^* E) = \mu^*(K_{X'} + \Delta_{X'}) + \mu^* \delta f'^* E$$

is lc over the generic point of E which in turn implies that

$$\pi^*(K_X + B) + \delta \mu^* f'^* E$$

is also lc over the generic point of E . Therefore, the coefficient of E in $B_{Z'}$ is at most $1 - \delta$. This gives a contradiction. Thus, every coefficient of \mathcal{B}_Z is at most $1 - \delta$. \square

4. PROOF OF THE MAIN RESULTS

Proof. (of Theorem 1.3) Let (X, B) and $f: X \rightarrow Z$ be as in Conjecture $S_{d, \epsilon, \mathcal{P}}$. If f is birational, then $B_Z = f_* B$ and if we take $M_Z = 0$, then $K_X + B = f^*(K_Z + B_Z)$ and (Z, B_Z) is ϵ -lc. So, $\delta = \epsilon$ works in this case. We can then assume that f is not birational. By taking a \mathbb{Q} -factorialisation we may assume that X is \mathbb{Q} -factorial. The log general fibres may change but only by an isomorphism in

codimension one. Pick $\epsilon' \in (0, \epsilon)$. By Lemma 3.3, we can replace X, Z with projective varieties so that (X, B) and $f: X \rightarrow Z$ satisfy the assumptions of Conjecture $S_{d, \epsilon', \mathcal{P}}$ (note that ϵ is replaced by ϵ').

First assume that the coefficients of B are rational numbers. Let $\delta' > 0$ be the number given by Proposition 3.5 for the data $d, \epsilon', \mathcal{P}$, and pick some $\delta \in (0, \delta')$. Let $g: Z' \rightarrow Z$ be an Ambro model of $K_X + B$ and $f: X \rightarrow Z$ as defined in 2.6. Since Z' is an Ambro model, $M_{Z'}$ is nef and good, that is, it is the pullback of a nef and big \mathbb{Q} -divisor M_T via some contraction $\pi: Z' \rightarrow T$. We can write $M_T \sim_{\mathbb{Q}} A_T + L_T$ where A_T is an ample \mathbb{Q} -divisor and $L_T \geq 0$. Put $A_{Z'} = \pi^* A_T$ and $L_Z = \pi^* L_T$. Thus, $M_{Z'} \sim_{\mathbb{Q}} A_{Z'} + L_{Z'}$ where $A_{Z'}$ is semi-ample and $L_{Z'} \geq 0$. Perhaps after replacing Z' we may assume that $\text{Supp}(B_{Z'} + L_{Z'})$ has simple normal crossing singularities. By our choice of δ' , each coefficient of $B_{Z'}$ is $\leq 1 - \delta'$.

Since $A_T + L_T$ is nef and A_T is ample,

$$A_T + \frac{1}{a+1}L_T = \frac{a}{a+1}A_T + \frac{1}{a+1}(A_T + L_T)$$

is ample for any $a > 0$. So, perhaps after replacing $L_{Z'}$ with $\frac{a}{a+1}L_{Z'}$ and $A_{Z'}$ with $A_{Z'} + \frac{1}{a+1}L_{Z'}$ for some sufficiently small $a > 0$, we may assume that the coefficients of $L_{Z'}$ are sufficiently small. Thus, we can assume that the coefficients of $B_{Z'} + L_{Z'}$ are all $\leq 1 - \delta$. Perhaps after replacing $A_{Z'}$ we can also assume that $\text{Supp}(B_{Z'} + A_{Z'} + L_{Z'})$ has simple normal crossing singularities and that the coefficients of $B_{Z'} + A_{Z'} + L_{Z'}$ are $\leq 1 - \delta$ and that $A_{Z'} \geq 0$.

Let A_Z, L_Z be the pushdown of $A_{Z'}, L_{Z'}$ respectively. Since

$$K_{Z'} + B_{Z'} + A_{Z'} + L_{Z'} = g^*(K_Z + B_Z + A_Z + L_Z)$$

we deduce that $(Z, B_Z + A_Z + L_Z)$ is a δ -lc pair. By putting $M_Z := A_Z + L_Z$ we finish the proof of the theorem when B has rational coefficients.

Now we come to the general case, that is, when the coefficients of B are not necessarily rational. We will do an approximation to reduce to the rational case. Pick $\epsilon'' \in (0, \epsilon')$, let $\delta'' > 0$ be the number given by Proposition 3.5 for the data $d, \epsilon'', \mathcal{P}$, and pick some $\delta \in (0, \delta'')$. Then, using Shokurov's polytopes [19], we can find real numbers $r_i \geq 0$ and rational boundaries B^i such that

- $\sum r_i = 1$ and $K_X + B = \sum r_i(K_X + B^i)$,
- $K_X + B^i \sim_{\mathbb{Q}} 0/Z$,
- $\text{Supp } B^i = \text{Supp } B$,
- (X, B^i) are ϵ'' -lc.

Applying the arguments above in the rational case, for each i we can choose $M_Z^i \geq 0$ such that

$$K_X + B^i \sim_{\mathbb{Q}} f^*(K_Z + B_Z^i + M_Z^i)$$

and that $(Z, B_Z^i + M_Z^i)$ is a δ -lc pair.

Fix a prime divisor D on Z and let its coefficient in B_Z and B_Z^i be b and b^i respectively. By definition, the pair $(X, B^i + (1 - b^i)f^*D)$ is lc over the generic point of D . But then the pair $(X, B + \sum r_i(1 - b^i)f^*D)$ is lc over the generic

point of D . This means that

$$1 - \sum r_i b^i = \sum r_i (1 - b^i) \leq 1 - b$$

which in turn implies that $\sum r_i b^i \geq b$. In other words, $\sum r_i B_Z^i \geq B_Z$.

Now, the pair

$$(Z, \sum r_i B_Z^i + \sum r_i M_Z^i)$$

is δ -lc and by putting

$$M_Z := \sum r_i B_Z^i + \sum r_i M_Z^i - B_Z$$

we conclude that $(Z, B_Z + M_Z)$ is δ -lc and

$$K_X + B \sim_{\mathbb{R}} f^*(K_Z + B_Z + M_Z)$$

This completes the proof of the theorem. \square

We need the following theorem for the proof of Corollary 1.4.

Theorem 4.1. *Let $\epsilon, \lambda > 0$ be real numbers. Consider the set of pairs (F, B_F) introduced just before Corollary 1.4 with the extra assumption $d \leq 3$. Then, \mathcal{R} , the set of the couples $(F, \text{Supp } B_F)$ is a bounded family.*

Proof. First assume that $\dim F = 1$ which means that $F \simeq \mathbb{P}^1$. Replacing λ with a smaller number we may assume that it is rational. Since $\deg B_F = 2$ and each coefficient of B_F is $\geq \lambda$, the number of components of $\text{Supp } B_F$ is bounded only depending on λ . Then, the set of the couples $(F, \text{Supp } B_F)$ belongs to a bounded family using Hilbert schemes of zero-dimensional subschemes of F .

Now assume that $\dim F = 2$. Let Δ_F be the boundary obtained from B_F by replacing each coefficient with λ . By definition, $C_F := B_F - \Delta_F \geq 0$. Pick a small number $t > 0$ so that $(F, B_F + tC_F)$ is still klt. Run the LMMP on $K_F + B_F + tC_F \sim_{\mathbb{R}} tC_F$. We get a log minimal model on which the pushdown of C_F is nef. By replacing F with that model we could assume that C_F is nef. Note that the LMMP can contract only the components of B_F so we can pullback boundedness to the original setting.

By Alexeev [1] the varieties F belong to a bounded family. In particular, the Cartier index of $K_F + \Delta_F$ is bounded. So, we can pick a \mathbb{Q} -divisor $A_F \geq 0$ such that $K_F + \Delta_F + A_F \sim_{\mathbb{Q}} 0$ and such that the coefficients of $\Delta_F + A_F$ belong to a fixed finite set depending only on ϵ and λ . Now, apply [10, Corollary 1.7] to get the boundedness of the couples $(F, \text{Supp } \Delta_F) = (F, \text{Supp } B_F)$. \square

Proof. (of Corollary 1.4) This follows from Theorems 4.1 and 1.3. \square

Remark 4.2 Actually, if one tries to prove the boundedness of \mathcal{R} for $d > 3$ inductively, then variants of Theorem 1.3 appear naturally in the induction process. More precisely: pick (F, B_F) as defined just before Corollary 1.4. After running a suitable LMMP one can assume that F has a Mori fibre space structure $F \rightarrow G$. Assume that $\dim G > 0$. The couples associated to the log general fibres of this morphism are bounded by induction. One uses a variant of Theorem 1.3 to show that G is also bounded (one needs a stronger form of 1.3 in which the coefficients of $B_G + M_G$ are not too small and this needs an effective

version of Ambro's result on M_G); the next step is to use these boundedness results to prove that F itself together with $\text{Supp } B_F$ are bounded. This is more related to the work of Shokurov mentioned in the introduction. On the other hand, if $\dim G = 0$, one needs different arguments.

Proof. (of Corollary 1.5) This is immediate by Theorem 1.3 and [10, Corollary 1.7]. \square

Proof. (of Corollary 1.6) Let (X, B) and $f: X \rightarrow Z$ be as in Conjecture $S_{d,\epsilon,\mathcal{P}}$ such that $d \leq \dim Z + 1$. We may assume that $d = \dim Z + 1$ otherwise f is birational and we can argue as in the proof of Theorem 1.3. Using the arguments of the proof of Lemma 3.2, Proposition 3.5, and Theorem 1.3, we can reduce the problem to the case $\dim Z = 1$ and $\dim X = 2$. Moreover, we only need to show that the coefficients of B_Z are $\leq 1 - \delta$ for a fixed $\delta > 0$ depending only on ϵ .

By replacing X with its minimal resolution, we may assume that X is smooth. Next by running an LMMP/ Z on K_X we can assume that $X \rightarrow Z$ is an extremal contraction, that is, in this case a \mathbb{P}^1 -bundle. Fix $\epsilon' \in (0, \epsilon)$. Pick a closed point $D \in Z$ and let t be the ϵ' -lc threshold of f^*D with respect to the pair (X, B) , that is, t is the largest number so that $(X, B + tf^*D)$ is ϵ' -lc. Here $T := f^*D$ is a reduced curve. If the coefficient of T in $B + tf^*D$ is $1 - \epsilon'$, then $t \geq (\epsilon - \epsilon')$ because the coefficient of T in B is at most $1 - \epsilon$. In this case, the coefficient of D in B_Z is at most $1 - \epsilon + \epsilon'$ and $\delta = \epsilon - \epsilon'$ works. From now on we assume that the coefficient of T in $B + tf^*D$ is $< 1 - \epsilon'$.

There is a prime exceptional/ X divisor E such that $a(E, X, B + tf^*D) = \epsilon'$. Let $Y \rightarrow X$ be the extremal contraction which extracts E . There is another extremal ray on Y/Z which we can contract to get X' , and $X' \rightarrow Z$ is an extremal contraction. However, X' may not be smooth. Let g, h, f' denote the contractions $Y \rightarrow X$, $Y \rightarrow X'$, and $X' \rightarrow Z$ respectively. Write $K_{X'} + B' = h_*g^*(K_X + B)$ and let $T' = \text{Supp } f'^*D$. Then, the coefficient of T' in $B' + tf'^*D$ is $1 - \epsilon'$. Since the coefficient of T' in B' is at most $1 - \epsilon$, it is enough to show that $f'^*D = m'T'$ where m' is bounded depending on ϵ .

Applying the boundedness of ϵ' -lc complements in dimension two [5], we get a real number $\epsilon'' > 0$ and a finite set Λ of rational numbers depending only on ϵ' which satisfy: there is a boundary Δ' such that (X', Δ') is ϵ'' -lc, $K_{X'} + \Delta' \sim_{\mathbb{Q}} 0/Z$, and the coefficients of Δ' belong to Λ . Now we can apply Corollary 1.4 to get the bound on m' . \square

Remark 4.3 An obvious question to ask is: what can we do about Conjecture $S_{d,\epsilon,\mathcal{P}}$ if \mathcal{P} is not bounded? Let (X, B) and $f: X \rightarrow Z$ be as in Conjecture $S_{d,\epsilon,\mathcal{P}}$, and let (F, B_F) be a log general fibre. In some cases, the couples $(F, \text{Supp } B_F)$ belong to a bounded family \mathcal{P}' even if \mathcal{P} is not bounded. For example, all the couples in \mathcal{P} of dimension $\leq d - 1$ may belong to a bounded family \mathcal{P}' , that is, the unbounded part of \mathcal{P} may not be relevant to the conjecture. In this case, Conjecture $S_{d,\epsilon,\mathcal{P}'}$ implies Conjecture $S_{d,\epsilon,\mathcal{P}}$, and we can use Theorem 1.3 to prove Conjecture $S_{d,\epsilon,\mathcal{P}'}$.

In general, we cannot shrink \mathcal{P} to a bounded family \mathcal{P}' . The idea then is to modify the pair (X, B) to get boundedness. For example, as in the proof of Corollary 1.6, one can hope to find a real number $\epsilon' > 0$ and a fixed finite set Λ of real numbers depending only on d, ϵ' such that: there is a boundary Δ so that (X, Δ) is ϵ' -lc, $K_X + \Delta \sim_{\mathbb{Q}} 0/Z$, and the coefficients of Δ belong to Λ . Next one applies Corollary 1.5. This is closely related to the theory of complements [20][5].

REFERENCES

- [1] V. Alexeev; *Boundedness and K^2 for log surfaces*. Internat. J. Math. 5 (1994), no. 6, 779–810.
- [2] V. Alexeev, A. Borizov; *On the Log Discrepancies in Mori Contractions*. arXiv:1208.3271v1.
- [3] F. Ambro; *The moduli b -divisor of an lc-trivial fibration*. Compos. Math. 141 (2005), no. 2, 385–403.
- [4] F. Ambro; *Basic properties of log canonical centers*. arXiv:math/0611205v1.
- [5] C. Birkar; *Boundedness of epsilon-lc complements on surfaces*. Available at <https://www.dpmms.cam.ac.uk/~cb496/research.html>.
- [6] C. Birkar; *Existence of log canonical flips and a special LMMP*. Publ. Math. de l’IHÉS, Volume 115, Issue 1 (2012), 325–368.
- [7] C. Birkar, P. Cascini, C. D. Hacon, J. McKernan; *Existence of minimal models for varieties of log general type*. J. Amer. Math. Soc. 23 (2010), 405–468.
- [8] O. Fujino, Y. Gongyo; *On canonical bundle formulae and subadjunctions*. To appear in Michigan Math. J. arXiv:1009.3996v1.
- [9] C. D. Hacon, J. McKernan, C. Xu; *On the birational automorphisms of varieties of general type*. arXiv:1011.1464v1, To appear, Ann. of Math.
- [10] C. D. Hacon, J. McKernan, C. Xu; *ACC for log canonical thresholds*. arXiv:1208.4150v1.
- [11] Y. Kawamata; *Minimal models and the Kodaira dimension of algebraic fiber spaces*. J. Reine Angew. Math. 363 (1985), 1–46.
- [12] Y. Kawamata; *Subadjunction of log canonical divisors for a variety of codimension 2*, Contemporary Mathematics 207 (1997), 79–88.
- [13] Y. Kawamata; *Subadjunction of log canonical divisors, II*, Amer. J. Math. 120 (1998), 893–899.
- [14] J. Kollár, S. Mori; *Birational Geometry of Algebraic Varieties*. Cambridge University Press (1998).
- [15] L. Ein, R. Lazarsfeld, M. Mustață, M. Nakamaye, M. Popa; *Asymptotic invariants of line bundles*. Pure Appl. Math. Q. 1 (2005), 379–403.
- [16] S. Mori and Yu. Prokhorov; *On Q -conic bundles*. Publ. Res. Inst. Math. Sci. 44 (2008), no. 2, 315–369.
- [17] S. Mori and Yu. Prokhorov; *Multiple fibers of del Pezzo fibrations*. arXiv:0806.3809v1.
- [18] V.V. Shokurov; *Three-dimensional log flips*. With an appendix in English by Yujiro Kawamata. Russian Acad. Sci. Izv. Math. 40 (1993), no. 1, 95–202.
- [19] V.V. Shokurov; *3-fold log models*. Algebraic geometry, 4. J. Math. Sci. 81 (1996), no. 3, 2667–2699.
- [20] V.V. Shokurov; *Complements on surfaces*. Algebraic geometry, 10. J. Math. Sci. (New York) 102 (2000), no. 2, 3876–3932.
- [21] V.V. Shokurov; *Prelimiting flips*. Proc. Steklov Inst. Math. 240 (2003), 75–213.

DPMMS, Centre for Mathematical Sciences,
Cambridge University,
Wilberforce Road,
Cambridge, CB3 0WB,
UK
email: c.birkar@dpmms.cam.ac.uk